Definition 9.1

A group G is cyclic if there is an element $a \in G$ such that

$$G = \{a^n \mid n \in \mathbb{Z}\}$$

or, in other notation, $G = \langle a \rangle$. In such case we say that a is a *generator* of G.

Theorem 9.2

If G is a finite group then G is cyclic if and only if there is an element $a \in G$ such that $|a| = |G|$.

Theorem 9.3

Every subgroup of a cyclic group is cyclic.

Theorem 9.4

If G is a finite cyclic group and $H \subseteq G$ is a subgroup then $|H|$ divides $|G|$.

Theorem 9.5

If G is a finite cyclic group and $d > 0$ is an integer that divides $|G|$ then there exists exactly one subgroup $H \subseteq G$ such that $|H| = d$.

Theorem 9.6

Let $G = \langle a \rangle$ be a cyclic group of order n. An element a^k is a generator of G (i.e. $\langle a^k \rangle = G$) if and only if $\gcd(n, k) = 1$.

Exercise. In the group $\mathbb{Z}_{15}$ find all elements a such that a generates $\mathbb{Z}_{15}$

Theorem 9.7

Let $G_1 = \langle a_1 \rangle$ and $G_2 = \langle a_2 \rangle$ be finite cyclic groups. The group $G_1 \times G_2$ is cyclic if and only if $\gcd(|G_1|, |G_2|) = 1$.

Theorem 9.8

For $i = 1, \ldots, n$ let $G_i = \langle a_i \rangle$ be a cyclic group. The group $G_1 \times G_2 \times \ldots \times G_n$ is cyclic if and only if $\gcd(|G_i|, |G_j|) = 1$ for all $i \neq j$.