Exponentiation

Properties of exponentiation

Definition 6.1

Let G be a group. An *order* of an element $g \in G$ is the smallest integer $n \geq 1$ such that $g^n = e$. We write: $|g| = n$.

If $g^n \neq e$ for all $n \geq 1$ then we say that g is an element of an *infinite order* and we write $|g| = \infty$.

Exercise. Recall that the multiplication table of the dihedral group D_4 is as follows:

$\circ$	I	R_{90}	R_{180}	R_{270}	H	V	D	D'
I	I	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_{90}	R_{90}	R_{180}	R_{270}	I	D'	D	H	V
R_{180}	R_{180}	R_{270}	I	R_{90}	V	H	D'	D
R_{270}	R_{270}	I	R_{90}	R_{180}	D	D'	V	H
H	H	D	V	D'	I	R_{180}	R_{90}	R_{270}
V	V	D'	H	D	R_{180}	I	R_{270}	R_{90}
D	D	H	D'	V	R_{270}	R_{90}	I	R_{180}
D'	D'	V	D	H	R_{90}	R_{270}	R_{180}	I

Find the order of every element of D_4.

Exercise. Find the order of every element in the group $\mathbb{Z}_6$.

Theorem 6.2

If G is a finite group and $g \in G$ then $|g| < \infty$.

Theorem 6.3

If G is a group, $g \in G$ and $n \geq 1$ is an integer such that $g^n = e$, then $|g|$ divides n.

Theorem 6.4

If G is a group, and $a, b \in G$ are elements such that $|a|, |b| < \infty$ and $ab = ba$ then $|ab|$ divides $|a| \cdot |b|$.

Theorem 6.5

If G is a group, and $a \in G$ is element such that $|a| = n < \infty$ then

$$|a^k| = \frac{n}{\gcd(n, k)}$$

Exercise. Compute the order of the element $6 \in \mathbb{Z}_{10}$.