Definition 22.1

Let R be an integral domain. An element $a \in R$ is *irreducible* if $a \neq 0$, a is not a unit and if $a = bc$ for some $b, c \in R$ then either b or c is a unit.

Theorem 22.2

If R is an integral domain, $a \in R$ is irreducible and $u \in R$ is a unit then ua is irreducible.

Definition 22.3

Let R be an integral domain. Elements $a, b \in R$ are *associates* if $a = ub$ for some unit $u \in R$. We write: $a \sim b$.

Definition 22.4

A *unique factorization domain* (*UFD*) is an integral domain R that satisfies the following conditions:

1) if $a \in R$ is a non-zero, non-unit element then

$$a = b_1 \cdot \ldots \cdot b_k$$

for some irreducible elements $b_1, \ldots, b_k \in R$

2) if $b_1, \ldots, b_k$, $c_1, \ldots, c_l$ are irreducible elements such that

$$b_1 \cdot \ldots \cdot b_k = c_1 \cdot \ldots \cdot c_l$$

then $k = l$ and for some permutation $\sigma\colon \{1, \ldots, k\} \to \{1, \ldots, k\}$ we have $b_1 \sim c_{\sigma(1)}$, …, $b_k \sim c_{\sigma(k)}$.

Theorem 22.5

The ring $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.