Definition 17.1

A *ring* is set R equipped with two binary operations:

- *addition*, denoted $a + b$
- *multiplication*, denoted $a \cdot b$

satisfying the following properties:

1) R taken with addition is an abelian group (with the identity element $0 \in R$).

2) Multiplication is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any $a, b, c \in R$.

3) For any $a, b, c \in R$ we have $(a + b)c = ac + bc$ and $a(b + c) = ab + ac$.

Definition 17.2

We say that a ring R is *commutative* if $ab = ba$ for any $a, b \in R$.

We say that R is a *ring with unity* if there is an element $1 \in R$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.

Theorem 17.3

If a ring R has a unity, the the unity is unique.

Theorem 17.4

If a ring R. For any $a, b \in R$ we have:

1) $0 \cdot a = a \cdot 0 = 0$.

2) $a(-b) = (-a)b = -(ab)$.

3) $(-a)(-b) = ab$

4) if R has a unity $1 \in R$ then $(-1)a = a(-1) = -a$.

Definition 17.5

Let R be a ring. A *subring* of R is a subset $S \subseteq R$ such that S is a ring with respect to the addition and multiplication in R.

Theorem 17.6

Let R be a ring. A subset $S \subseteq R$ is a subring of R if and only if the following conditions are satisfied:

1) $0 \in S$

2) if $a, b \in S$ then $a + b \in S$ and $ab \in S$

3) if $a \in S$ then $(-a) \in S$

Definition 17.7

The *direct product* of rings R_1, R_1 is a ring $R_1 \times R_2$ defined as follows:

- Elements of $R_1 \times R_1$ are ordered tuples (a_1, a_2) where $a_i \in R_i$
- Addition and multiplication are given by

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1,\ a_2 + b_2)$$
$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1,\ a_2 b_2)$$