Theorem 16.1

If G is a finite abelian group then G is isomorphic to a direct product of cyclic groups whose orders are powers of primes:

$$G \cong \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \ldots \times \mathbb{Z}_{p_k^{r_k}}$$

for primes $p_1, \ldots, p_k$ and integers $r_1, \ldots, r_k \geq 1$ such that $p_1^{r_1} \cdot p_2^{r_2} \cdot \ldots \cdot p_k^{r_k} = |G|$.

Definition 16.2

A *short exact sequence* of groups is a sequence group homorphisms

$$K \xrightarrow{i} G \xrightarrow{q} H$$

such that:

- i is 1-1
- q is onto
- $\mathrm{Im}(i) = \mathrm{Ker}(q)$

Theorem 16.3

Consider a short exact sequence

$$K \xrightarrow{i} G \xrightarrow{q} H$$

where K, G, H are abelian groups. Assume that there exists a homomorphism $s\colon H \to G$ such that $q \circ s(h) = h$ for all $h \in H$. Then $G \cong K \times H$.

Corollary 16.4

Consider a short exact sequence

$$K \xrightarrow{i} G \xrightarrow{q} H$$

where K, G, H are abelian groups. Assume that there exists a homomorphism $s\colon H \to G$ such that $q \circ s$ is an isomorphism. Then $G \cong K \times H$.

Theorem 16.5

Let G be a finite abelian group. Assume that $|G| = p^r m$ where p is a prime, $r \geq 1$ and m is a number which is not divisible by p. Then $G = K \times H$ where $|K| = p^r$ and $|H| = m$.

Lemma 16.6

Let G be a finite abelian group. Assume that there exists a prime p such that the order of each element $g \in G$ is a power of p. For $m \in \mathbb{Z}$ consider the function

$$f\colon G \to G$$

given by $f(g) = g^m$. If m is not divisible by p then f is a group isomorphism.

Corollary 16.7

If G is a finite abelian group and $|G| = p_1^{r_1} p_2^{r_2} \cdot \ldots \cdot p_k^{r_k}$ where $p_1, p_2, \ldots, p_k$ are distinct primes then

$$G = G_1 \times G_2 \times \ldots \times G_k$$

where $|G_i| = p_i^{r_i}$.

Theorem 16.8

If G is an abelian group such that $|G| = p^n$ for some prime p then G is a direct product of cyclic groups:

$$G \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}} \times \ldots \times \mathbb{Z}_{p^{k_m}}$$

for some $k_1, k_2, \ldots, k_m$.