Definition 15.1

Let G be a group and X be a set. Given a function

$$\mu\colon G \times X \to X$$

denote $g \cdot x := \mu(g, x)$. We say that μ is a *group action* of G on the set X if the following conditions are satisfied:

1) $(gh) \cdot x = g \cdot (h \cdot x)$ for any $g, h \in G$ and $x \in X$.
2) $e \cdot x = x$ for any $x \in X$.

Definition 15.2

Let $\mu\colon G \times X \to X$ be a group action.

- The *orbit* of an element $x \in X$ is the subset of X given by

$$\mathrm{Orb}(x) = \{gx \mid g \in G\}$$

- The *stabilizer* of an element $x \in X$ is the subset of G given by

$$\mathrm{Stab}(x) = \{g \in G \mid gx = x\}$$

Theorem 15.3

Let $\mu\colon G \times X \to X$ be a group action and let $x, y \in X$. Then:

1) $x \in \operatorname{Orb}(x)$.

2) Either $\operatorname{Orb}(x) = \operatorname{Orb}(y)$ or $\operatorname{Orb}(x) \cap \operatorname{Orb}(y) = \varnothing$.

3) $\operatorname{Orb}(x) = \operatorname{Orb}(y)$ if and only if $y = gx$ for some $g \in G$.

Corollary 15.4

If $\mu\colon G \times X \to X$ is a group action and X is a finite set, then

$$|X| = |\operatorname{Orb}(x_1)| + |\operatorname{Orb}(x_2)| + \cdots + |\operatorname{Orb}(x_m)|$$

where $\operatorname{Orb}(x_1), \operatorname{Orb}(x_2), \ldots, \operatorname{Orb}(x_m)$ are all different orbits of the action.

Theorem 15.5

Let $\mu\colon G \times X \to X$ be a group action and let $x \in X$.

1) $\operatorname{Stab}(x)$ is a subgroup of G.

2) If $y = gx$ then $\operatorname{Stab}(y) = g\operatorname{Stab}(x)g^{-1}$.

3) If G is a finite group then $|G| = |\operatorname{Orb}(x)| \cdot |\operatorname{Stab}(x)|$

Theorem 15.6 (Cauchy Theorem)

If G is a finite group and p is a prime that divides $|G|$ then there exists an element of order p in G.

Definition 15.7

If p is a prime number then a p-group is a finite group of order p^r for some $r \geq 0$.

Corollary 15.8

A finite group G is a p-group if and only if the order of every element of G is a power of p

Theorem 15.9

If G is a p-group then there exists an element $a \in G$ such that $a \neq e$ and $ag = ga$ for all $g \in G$.