Recall:

- A normal subgroup of a group G is a subgroup $H \subseteq G$ such that for every $a \in G$ and $h \in H$ we have $aha^{-1} \in H$.
- We write $H \triangleleft G$ to denote that H is a normal subgroup of G.
- If $f\colon G \to K$ is a group homomorphism, then $\mathrm{Ker}(f) \triangleleft G$.

Theorem 14.1

Let G be a group and let $H \subseteq G$ be a subgroup. Then the following conditions are equivalent:

1) $H \triangleleft G$

2) For any $a \in G$ we have $aHa^{-1} = H$ where $aHa^{-1} = \{aha^{-1} \mid h \in H\}$.

3) For any $a \in G$ we have $aH = Ha$.

Theorem 14.2

Let G be a group and $H \triangleleft G$. Let $a_1, a_2, b_1, b_2 \in G$ be elements such that $a_1H = a_2H$ and $b_1H = b_2H$. Then $(a_1b_1)H = (a_2b_2)H$.

Definition 14.3

Let G be a group and let $H \triangleleft G$. The *quotient group* G/H is defined as follows:

- Elements of G/H are left cosets aH of H in G.
- Group operation: $aH \cdot bH = (ab)H$.
- The identity element: the coset $eH = H$.
- The inverse of aH: $a^{-1}H$.

Example. Take the dihedral group D_4:

$\circ$	I	R_{90}	R_{180}	R_{270}	H	V	D	D'
I	I	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_{90}	R_{90}	R_{180}	R_{270}	I	D'	D	H	V
R_{180}	R_{180}	R_{270}	I	R_{90}	V	H	D'	D
R_{270}	R_{270}	I	R_{90}	R_{180}	D	D'	V	H
H	H	D	V	D'	I	R_{180}	R_{90}	R_{270}
V	V	D'	H	D	R_{180}	I	R_{270}	R_{90}
D	D	H	D'	V	R_{270}	R_{90}	I	R_{180}
D'	D'	V	D	H	R_{90}	R_{270}	R_{180}	I

Theorem 14.4 (First Isomorphism Theorem)

Let $f\colon G \to H$ be a homomorphisms of groups which is onto. Then

$$H \cong G/\mathrm{Ker}(f)$$

Corollary 14.5

For any normal subgroup K of a group G there exists a homomorphism $f\colon G \to H$ such that $\mathrm{Ker}(f) = K$.