Definition 11.1

Let G, H be groups. A group homomorphism is a function

$$f\colon G \to H$$

which for any $a, b \in G$ satisfies $f(a \cdot b) = f(a) \cdot f(b)$

Theorem 11.2

Let $f\colon G \to H$ be a groups homomorphism. Then:

- $f(e_G) = e_H$ where e_G and e_H are the identity elements in G and H, respectively.
- $f(a^{-1}) = f(a)^{-1}$ for any $a \in G$.

Examples.

Theorem 11.3

Let $f\colon G \to H$ be a homomorphism of groups and let $a \in G$. If $|a| < \infty$ then $|f(a)|$ divides $|a|$.

Definition 11.4

Let $f\colon G \to H$ be a group homomorphism. The *kernel of f* is the subset of G defined by

$$\mathrm{Ker}(f) = \{g \in G \mid f(g) = e\}$$

The *image of f* is the subset of H given by

$$\mathrm{Im}(f) = \{f(g) \mid g \in G\}$$

Theorem 11.5

If $f\colon G \to H$ is a homomorphism of groups then $\mathrm{Ker}(f)$ is a subgroup of G and $\mathrm{Im}(f)$ is a subgroup of H.

Examples.

Theorem 11.6

If $f\colon G \to H$ is a homomorphism then $f(a) = f(b)$ if and only if $b = ak$ for some $k \in \operatorname{Ker}(f)$.

Corollary 11.7

A homomorphism of groups $f\colon G \to H$ is 1-1 if and only if $\operatorname{Ker}(f) = \{e\}$.

Corollary 11.8

If $f\colon G \to H$ is a homomorphism of groups, and $f(a) = b$ for some $a \in G$, $b \in H$ then

$$f^{-1}(b) = \{ak \mid k \in Ker(f)\ \}$$

Theorem 11.9

Let $f\colon G \to H$ is a homomorphism of groups then $g \in \mathrm{Ker}(f)$ if and only if for each $a \in G$ we have $aga^{-1} \in \mathrm{Ker}(f)$.

Definition 11.10

Let G be a group. We say that a subgroup $H \subseteq G$ is a *normal subgroup* of G if for any $h \in H$ and $g \in G$ we have $ghg^{-1} \in H$.

We write $H \triangleleft G$ to denote that H is a normal subgroup of G.

Corollary 11.11

If $f\colon G \to H$ is a homomorphism of groups then $\mathrm{Ker}(f)$ is a normal subgroup of G.

Example. Consider the dihedral group D_4:

$\circ$	I	R_{90}	R_{180}	R_{270}	H	V	D	D'
I	I	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_{90}	R_{90}	R_{180}	R_{270}	I	D'	D	H	V
R_{180}	R_{180}	R_{270}	I	R_{90}	V	H	D'	D
R_{270}	R_{270}	I	R_{90}	R_{180}	D	D'	V	H
H	H	D	V	D'	I	R_{180}	R_{90}	R_{270}
V	V	D'	H	D	R_{180}	I	R_{270}	R_{90}
D	D	H	D'	V	R_{270}	R_{90}	I	R_{180}
D'	D'	V	D	H	R_{90}	R_{270}	R_{180}	I