For full credit explain your reasoning, showing all relevant work.

Exercise 1. Let F be a field with 81 elements. Show that $5a = -a$ for each $a \in F$.

Exercise 2. A *Boolean ring* is a ring R with the property that $a^2 = a$ for all $a \in R$.

a) Show that if R is a Boolean ring then $a = -a$ for any $a \in R$.

b) Show that every Boolean ring is commutative.

Exercise 3. a) In the ring $\mathbb{Z}_{15}$, find two elements a, b that are zero divisors, but such that $a + b \neq 0$ and $a + b$ is not a zero divisor. Justify your answer.

b) In the ring $\mathbb{Z}_{15}$, find two elements a, b that are units, but such that $a + b \neq 0$ and $a + b$ is not a unit. Justify your answer.

PRACTICE PROBLEMS

Exercises below are for practice only - do not turn them in for grading.

Practice Exercise 1. a) Let R be an integral domain. Show that if $a \in R$ is an element such that $a^2 = 1$, then either $a = 1$ or $a = -1$.

b) Find all elements $a \in \mathbb{Z}_{12}$ such that $a^2 = 1$.

Practice Exercise 2. Let R be a a ring with unity $1 \in R$. An element $a \in R$ is *nilpotent* if $a^n = 0$ for some $n \geq 1$. Show that if a is a nilpotent element, then the element $1 - a$ is a unit in R.

Hint: Try $n = 2$ and $n = 3$ first, then generalize.

Practice Exercise 3. Show that every non-zero element of $\mathbb{Z}_n$ is either a unit or a zero divisor.

Practice Exercise 4. a) Describe all zero divisors in the ring $\mathbb{Z} \times \mathbb{Q}$.

b) Describe all units in $\mathbb{Z} \times \mathbb{Q}$.