For full credit explain your reasoning, showing all relevant work.

Exercise 1. We say that an action of a group G on a set X is *transitive* if for any $x, y \in X$ there is $g \in G$ such that $g \cdot x = y$.

Show that if a finite group G acts transitively on a finite set X then

$$\sum_{x \in X} |\operatorname{Stab}(x)| = |G|$$

Exercise 2. Let G be a group such that $|G| = p^r$ for some prime p and $r \geq 1$. Assume that G acts on a set X, and that $|X| = n$ where n is not divisible by p. Show that there is an element $x_0 \in X$ such that $gx_0 = x_0$ for all $g \in G$.

Exercise 3. Let G be an abelian group such that $|G| = 120$ and that G contains exactly 3 elements of order 2. Find a group H of the form

$$H = \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_k^{r_k}}$$

where $p_1, \ldots, p_k$ are (not necessarily different) prime numbers, such that $G \cong H$. Justify your answer.

PRACTICE PROBLEMS

Exercises below are for practice only - do not turn them in for grading.

Practice Exercise 1. Let G be a finite group acting on a finite set X. For $g \in G$ let $\mathrm{Fix}(g)$ be the set of elements of X which are fixed by g:

$$\mathrm{Fix}(g) = \{x \in X \mid gx = x\}$$

Show that

$$\sum_{g\in G} |\,\mathrm{Fix}(g)| = \sum_{x\in X} |\,\mathrm{Stab}(x)|$$

Practice Exercise 2. Let G be a finite group. Assume that $|G| = 45 = 5 \cdot 9$. The goal of this exercise is to show that there exists a subgroup $H \subseteq G$ such that $|H| = 9$.

Let X be the set of all subsets $A \subseteq G$ such that A consist of 9 elements. The group G acts on X as follows. If $A \in X$ and $A = \{a_1, a_2, \ldots, a_9\}$ then for $g \in G$ we define

$$g \cdot A = \{ga_1, ga_2, \ldots, ga_9\}$$

a) Show that $|X|$ (i.e. the number of elements of X) is not divisible by 3.

b) Show that there is $A_0 \in X$ such that $|\,\mathrm{Orb}(A_0)|$ is not divisible by 3.

c) Show that 9 divides $|\,\mathrm{Stab}(A_0)|$, where A_0 is as in part b).

d) Show that if $A_0 = \{a_1, a_2, \ldots, a_9\}$ and $g \in \mathrm{Stab}(A_0)$ then g must be of the form $a_i a_1^{-1}$ for some $1 \leq i \leq 9$. This shows that $|\,\mathrm{Stab}(A_0)| \leq 9$.

e) Conclude that $|\,\mathrm{Stab}(A_0)| = 9$ and so we can take $H = \mathrm{Stab}(A_0)$.

Practice Exercise 3. Show that, up to isomorphism, there are two abelian groups of order 108 that have exactly 2 elements of order 3.

Practice Exercise 4. Let $p_1, p_2, \ldots, p_{100}$ be different primes. Up to isomorphism, how many different abelian groups are there of order $p_1^4 \cdot p_2^4 \cdot \ldots \cdot p_{100}^4$? Justify your answer.