For full credit explain your reasoning, showing all relevant work.

Exercise 1. Let G be a group and $H \subseteq G$ be a subgroup. Show that if $[G : H] = 2$ then H is a normal subgroup of G.

Exercise 2. Let $\mathbb{Q}$ be a the group of rational numbers with addition. Show that for each $n = 1, 2, \ldots$ there is an element of order n in $\mathbb{Q}/\mathbb{Z}$.

Note: Since $\mathbb{Q}$ uses the additive notation, elements of $\mathbb{Q}/\mathbb{Z}$ are cosets of the form $a + \mathbb{Z}$ for $a \in \mathbb{Q}$.

Exercise 3. Recall the the center of a group G is a subgroup $Z(G)$ of G consisting of elements that commute with all elements in G:

$$Z(G) = \{a \in G \mid ab = ba \text{ for all } b \in G\}$$

a) Show that $\mathbb{Z}(G)$ is normal subgroup of G.

b) Let G be a group such the quotient group $G/Z(G)$ is cyclic. Show that G is abelian.

PRACTICE PROBLEMS

Exercises below are for practice only - do not turn them in for grading.

Practice Exercise 1. Let G be a group and let $H \triangleleft G$. Show if $g \in G$ is an element of finite order then the order of $gH \in G/H$ divides $|g|$.

Practice Exercise 2. Let G be a finite group, $H \triangleleft G$, and let $g \in G$. Show that if $\gcd(|g|, |H|) = 1$, then $|g| = |gH|$, where $|gH|$ denotes the order of the element $gH \in G/H$.

Practice Exercise 3. a) Let $\mathbb{Q}$ be a the group of rational numbers with addition. Show that if $H \subseteq \mathbb{Q}$ is any subgroup such that $H \neq \mathbb{Q}$ then the quotient group $\mathbb{Q}/H$ contains infinitely many elements.

Note: Since $\mathbb{Q}$ uses the additive notation, elements of $\mathbb{Q}/H$ are cosets of the form $a + H$ for $a \in \mathbb{Q}$.

b). Let G be a finite group consisting of more than one element. Show that there does not exist a homomorphism $f\colon \mathbb{Q} \to G$ which is onto.

Practice Exercise 4. Let G be an abelian group and let $T \subseteq G$ be the set of all elements of G that are of a finite order.

a) Show that T is a subgroup of G.

b) Show that every non-identity element of the quotient group G/T has infinite order.