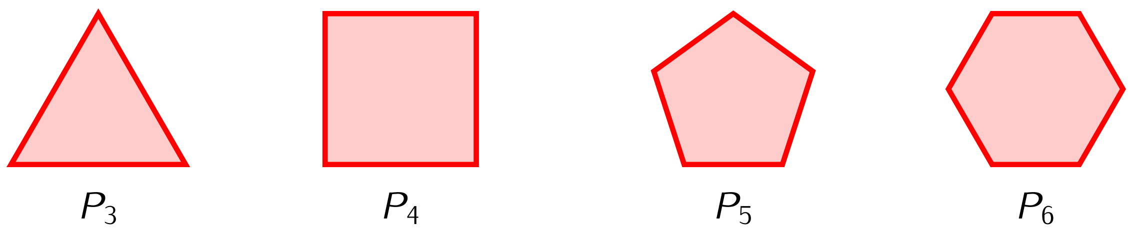
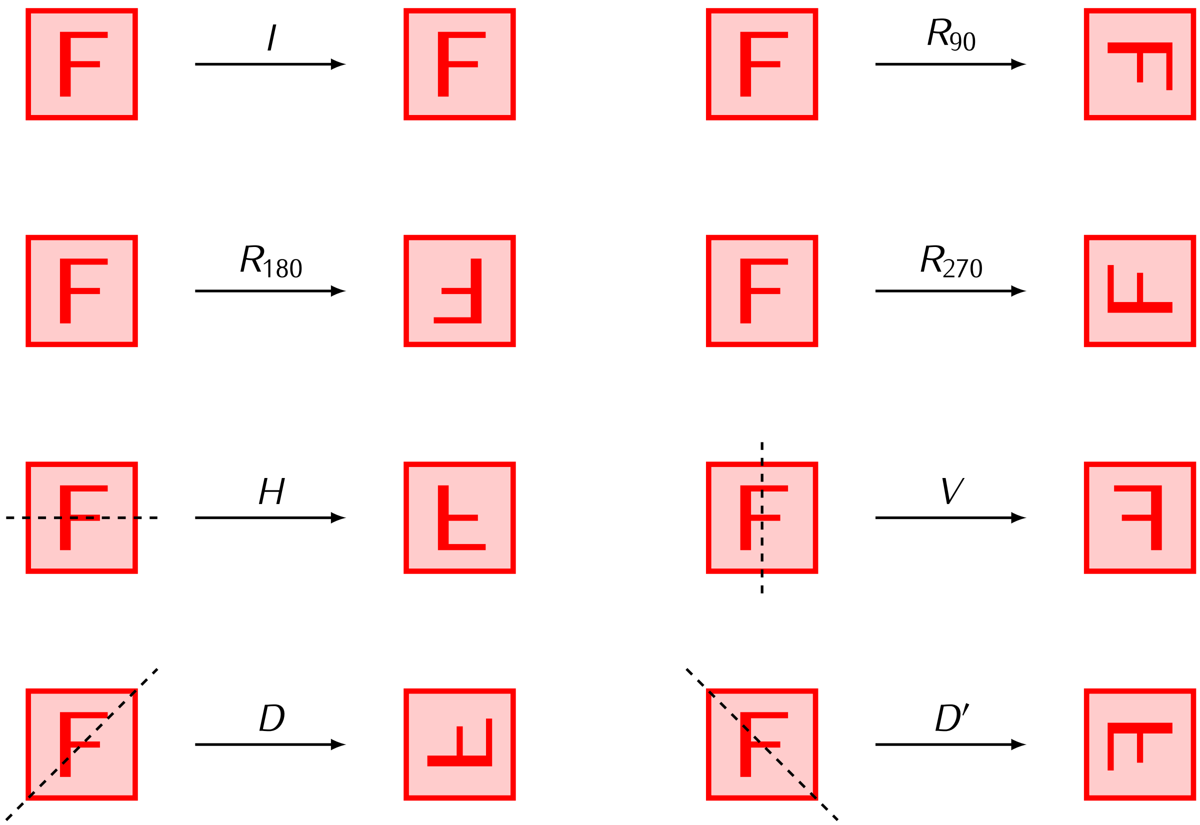
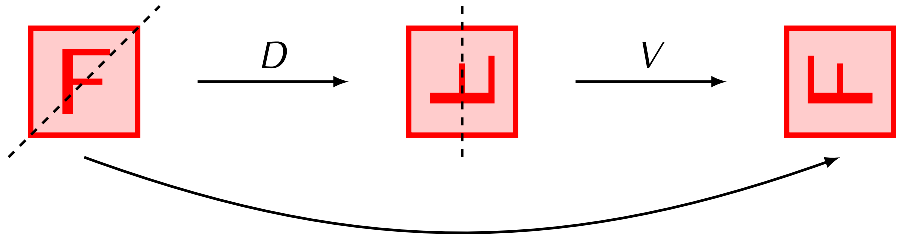
Regular polygons P_n with n sides:



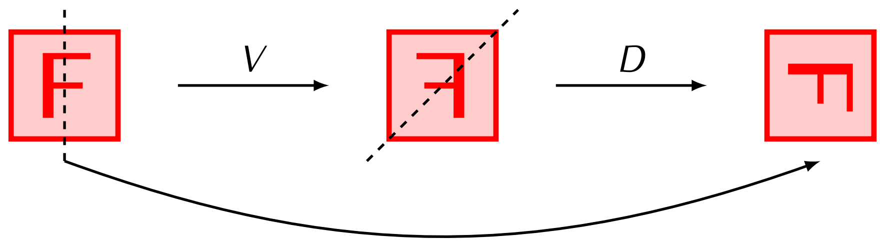
Symmetries of P_4:



Composition of symmetries:



$V \circ D = R_{270}$



$D \circ V = R_{90}$

Composition table of symmetries of a square:

$\circ$	I	R_{90}	R_{180}	R_{270}	H	V	D	D'
I	I	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_{90}	R_{90}	R_{180}	R_{270}	I	D'	D	H	V
R_{180}	R_{180}	R_{270}	I	R_{90}	V	H	D'	D
R_{270}	R_{270}	I	R_{90}	R_{180}	D	D'	V	H
H	H	D	V	D'	I	R_{180}	R_{90}	R_{270}
V	V	D'	H	D	R_{180}	I	R_{270}	R_{90}
D	D	H	D'	V	R_{270}	R_{90}	I	R_{180}
D'	D'	V	D	H	R_{90}	R_{270}	R_{180}	I

For $n \geq 3$ the dihedral group D_n is defined as follows:

- **Elements of D_n:** symmetries of the regular polygon with n sides.
- **Group operation:** Composition of symmetries (e.g. $V \circ D = R_{270}$).
- **The identity element:** The identity symmetry I.

Definition 5.1

The *order* of a group G is the number of elements of G. It is denoted by $|G|$. If G has infinitely many elements, we write $|G| = \infty$.

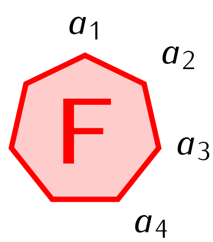
Examples.

- $|D_4| = 8$
- $|\mathbb{Z}_n| = n$
- $|\mathbb{Z}| = \infty$
- $|\mathbb{Q}| = \infty$
- $|\mathbb{R}| = \infty$
- $|GL(n, \mathbb{R})| = \infty$

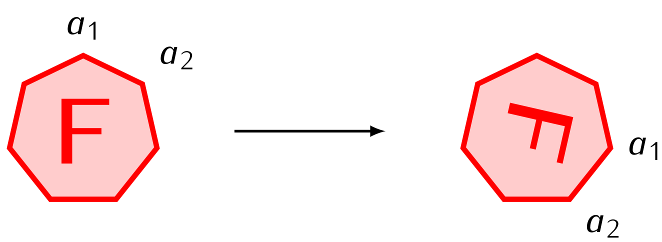
Theorem 5.2

For $n \geq 3$ we have $|D_n| = 2n$.

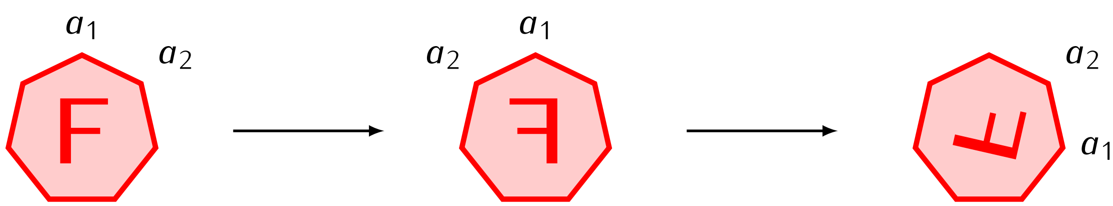
Proof. Let P_n be a regular polygon with vertices $a_1, a_2, \ldots, a_n$:



Each symmetry of P_n is uniquely determined once we know where is sends vertices a_1 and a_2. For every $1 \leq i \leq n$ there is a symmetry that sends a_1 to a_i and a_2 to a_{i+1} given by a rotation:



Also, for every i we there is a symmetry that sends a_1 to a_i and a_2 to a_{i-1}. This is given by a composition of a reflection with respect to a line that passes through a_1 and a rotation:

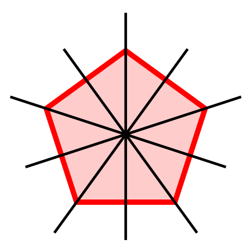


Altogether, this gives $2n$ possible symmetries of P_n. □

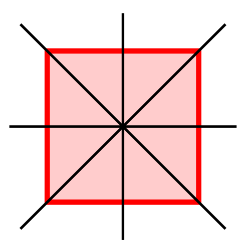
Note. The dihedral group D_n consists of the following elements:

1) n rotations by the angles of $k \cdot \frac{360}{n}$ degrees for $k = 0, \ldots, n-1$. For $k = 0$ this gives the rotation by 0 degrees, i.e. the identity symmetry.

2) n reflections with respect to different symmetry axes. If n is odd, there is one symmetry axis for each vertex of the polygon P_n:



If n is even, there are $\frac{n}{2}$ symmetry axes passing through pairs of opposite vertices and $\frac{n}{2}$ symmetry axes crossing opposite sides of P_n:



Definition 5.3

Let G be a group, and let $S \subseteq G$ be a subset of G. We say that the set S *generates* G if every element of G can be obtained as a product of some elements of S and inverses of elements of S.

Examples.

- Let P_n be a regular polygon with vertices $a_1, \ldots, a_n$. In the proof of Theorem 5.2 we have seen that every symmetry of P_n can be obtained by composing some rotation of P_n and a reflection D with respect to the line that passes through the vertex a_1. Moreover, every rotation of P_n can be obtained by composing some number of times the rotation R by the angle of $\frac{360}{n}$ degrees. Thus, every symmetry of P_n can be obtainted as some product of R and D. This means that the set $\{R, D\}$ generates the dihedral group D_n.

- The group of integers $\mathbb{Z}$ is generated by a set $\{1\}$ consisting of single element $1 \in \mathbb{Z}$, since every element of $\mathbb{Z}$ can be obtained by adding some number of times 1 and -1.

- The group of integers $\mathbb{Z}_n$ is generated by a set $\{1\}$ consisting of single element $1 \in \mathbb{Z}$.

- The set $\{2\}$ generates $\mathbb{Z}_3$, but it does not generate $\mathbb{Z}_4$.